Exam 9 Cookbook

2022 Sitting

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RisingFellow

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Published By:

Rising Fellow
United States, NY, 14610
www.RisingFellow.com

Contact: info@RisingFellow.com

Published in United States

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Introduction

The purpose of the Exam 9 Cookbook is to prepare you to confidently answer calculation-based problems on exam day without wasting time trying to "think through" a problem-solving approach before starting the solution.

Since the 2016 sitting, hundreds of actuaries have used the Exam 7, Exam 8, and Exam 9 Cookbooks to help them pass and get one step closer to their FCAS.

My goal with Rising Fellow is to help you prepare for the exam with less frustration so that you have your best exam sitting yet!

The Structure

The Exam 9 Cookbook goes through the different calculation-based problem-types that I believe are reasonably testable based on the syllabus. By exam day, you should know how to solve each one.

Inside, you'll find a separate section for each testable problem-type. Each section has the following structure:

Original Practice Problem

Each section has an original practice problem that demonstrates the problem-type. I wrote these based off of the syllabus papers to have a similar difficulty-level and style to what you might see on an exam.

Solution Recipe

The solution recipe solves the practice problem from start to finish and shows the step-by-step approach you should take to answer a similar problem. For each step, you'll see:

- The description for what to do in the step
- The formula(s) necessary for the step
- The formula(s) translated from symbolic notation to plain-English
- Calculations for the step to solve the example problem

Discussion

Each section includes discussion to add clarity about the problem-solving approach. The discussion also covers underlying concepts that might come up on a part b or part c essay question.

For many problems, I point out potential "twists" that could show up on the exam that would make an exam problem more difficult. Since you've taken actuarial exams up to this point, you know that straightforward exam problems are more the exception than the rule!

Source

Each section references the pages in the syllabus reading that you can cross-reference for more information and details.

More Practice

Here, you'll see references to past CAS problems and problems from the Exam 9 Problem Pack. You'll find this helpful especially closer to the exam if there are particular types of problems that you are struggling with. This section includes references to CAS problems since the 2011 exam, which are the past exams in the current syllabus structure.

How to Best Use the Exam 9 Cookbook

Below is a suggested guide for how you can incorporate the Exam 9 Cookbook in your own study schedule along with the syllabus material and a typical study manual. This is the general approach that I used when I took my fellowship exams.

For each of those exams I had a main study manual as well as the Exam Cookbook, which I built out while I studied for the exam (but you don't need to waste time doing that part!)

First pass through the syllabus

While you're reading a particular paper in the syllabus and your main study manual to learn the material, use the Exam 9 Cookbook to clearly identify what problem-types you need to know from the paper. Study the steps in the solution recipe to learn how to solve the problem-types.

Second pass through the syllabus

Review the steps for the problem-types and make sure you have an intuitive understanding of how to solve the problem. If you also purchased the Exam 9 Problem Pack, start working the practice problems from it as well as past CAS problems.

The first level of understanding is to be able to follow the recipe and nod your head as you read the steps. The next level of understanding is to be able to apply the steps to solve a problem with a blank answer sheet in front of you. At this point, your goal is to begin to build that level of understanding.

Review and Practice Problems (around 6 weeks to 2 weeks before the exam)

At this point you should have a good understanding of the syllabus and how to use the recipe steps to systematically solve the different calculation problems. During this period, you should be doing lots of problems across the syllabus and targeting problem-types that you are finding particularly challenging.
By the end of this phase, you might not have all the formulas memorized, but you should know all the steps and how to apply them to solve problems without needing to think too much before beginning the solution.

You also should continue building your understanding of the concepts and preparing for essay and Blooms Taxonomy questions. I found it helpful to create flashcards from the papers as well as to re-read sections of the syllabus papers that appear to be likely sources of essay problems.

Final Weeks

In the final weeks, focus on taking practice exams to see problems from the entire syllabus. When taking practice exams, work on your exam strategy to make sure you're able to finish the exam and maximize your points.

Prepare for essay problems in the final weeks by using flashcards to make sure that you know all the details necessary.

Prepare for calculation problems by reviewing the recipes in the Exam 9 Cookbook in a similar fashion to how you use flashcards for essay problems. Using this approach on my fellowship exams, I was able to rapidly review the steps and formulas for how to solve each problem-type that might show up on the exam. This was a huge benefit and gave me a lot of confidence going into the exam.

Exam Day

I used the original Exam Cookbooks together with a traditional study manual using the approach above to take my fellowship exams. On exam day, for almost every calculation problem I was able to start writing the solution without wasting time trying to think through how to solve the problem. I had an intuitive understanding of how to solve each of the problems following the step-by-step recipes.

If you follow this approach, you should be able to develop a similar level of understanding and confidence going into the exam room.

Excel Version for Computer-Based Testing Preparation

For each recipe, there is an accompanying Excel version. Make sure to review those so that you know how to solve problems in the Excel format.

Errata

I always hated seeing errors in study manuals when I studied for exams, so I make every effort to ensure the study materials are accurate. Nevertheless, there may still be some errors in the final version, so I keep an updated errata. Please make sure to check it regularly for any fixes. The link is below:

https://risingfellow.com/errata

Getting Unstuck

The syllabus material on the exam is challenging. If you get stuck on a section in the cookbook, I highly recommend first cross-referencing the cookbook section with the syllabus paper. Often that will answer your question quickly and will help you better see the connections between different exam concepts.

If you still are stuck on a section or if you see an error, you can send an email to **exam9@RisingFellow.com**. I will do my best to get back to you as soon as I'm able to.

Investor Utility and Indifference Curve

BKM 6

Problem

Two different risk-averse investors have degrees of risk aversion $A_1 = 2$ and $A_2 = 4$. They are evaluating which of two different portfolios to invest in. The portfolios have the following risk-return characteristics:

Low-Risk Portfolio:	$E(r) = .06$	$\sigma = .04$
High-Risk Portfolio:	$E(r) = .12$	$\sigma = .18$

The utility function is $U = E(r) - \tfrac{1}{2} A\sigma^2$

a. Determine which portfolio each investor should select.

b. Graph the indifference curve for investor 1 and the Low-Risk Portfolio. Describe how the corresponding indifference curve for investor 2 would relate to it.

c. Determine the range of return the risk-free rate would need for investor 2 to select the risk-free asset over the selected portfolio from part a.

Solution Recipe

Part a – Calculating Utility Values

1) **Calculate the utility of each portfolio based on the investor's degree of risk aversion.**

$$\boxed{U = E(r) - \tfrac{1}{2} A\sigma^2}$$

Note:
The utility function formula could be different.

$$U_{1,Low-Risk} = .06 - \tfrac{1}{2} \times 2 \times .04^2 = .0584$$

$$U_{1,High-Risk} = .12 - \tfrac{1}{2} \times 2 \times .18^2 = .0876$$

$$U_{2,Low-Risk} = .0568$$

$$U_{2,High-Risk} = .0552$$

2) **Select the portfolio for each investor with the highest utility.**

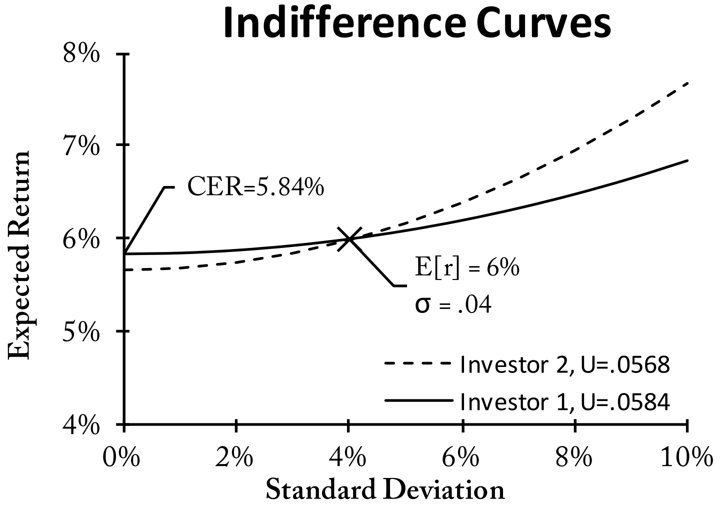
Investor 1 should select the **high-risk portfolio** and investor 2 should select the **low-risk portfolio**.

Part b – Indifference Curves

1) **Graph the indifference curve for an investor with the $\mathrm{E}\left[r_p\right]$ on the y-axis and σ_p on the x-axis.**

Key Points on the Graph

- Certainty Equivalent Rate – This is the *y*-axis intercept
- Portfolio E[r] and σ – Crossover point for the different investors



	Expected Return	
Std Dev	Investor 1	Investor 2
0.00	**5.84%**	**5.68%**
0.02	5.88%	5.76%
0.04	**6.00%**	**6.00%**
0.06	6.20%	6.40%
0.08	6.48%	6.96%
0.10	6.84%	7.68%

Investor 1:

$$.0584 = E(r)_{\sigma=.02} - \tfrac{1}{2} \times 2 \times .02^2$$

$$\rightarrow E(r)_{\sigma=.02} = 5.88\%$$

2) Describe the difference between the indifference curves of the two investors for the same portfolio

For an investor with a *higher* degree of risk-aversion, we should see:

- Lower *y*-axis intercept for the same portfolio
    - The certainty-equivalent rate will be lower for a more risk-averse investor.
- Steeper rise in the indifference curve
    - A more risk-averse investor will demand relatively higher levels of return as the level of risk (standard deviation) increases compared to a lower risk-aversion investor.
- The indifference curves will cross at the expected return and standard deviation of the portfolio.

Important Note: The utility value for this indifference curve is *lower* than the lower risk-aversion investor's indifference curve for the same portfolio (see Discussion for more).

Part c – Certainty-Equivalent Rate

3) Calculate the Certainty-Equivalent Rate as the expected return a risk-free asset would need to provide the same utility value as the risky asset.

$CER \rightarrow \mathrm{E}[r]$ with $\sigma^2 = 0$ providing the same U

$$.0568 = CER_{2,Low-Risk} - \tfrac{1}{2} \times 4 \times 0^2$$

$$\rightarrow CER_{2,Low-Risk} = \boxed{5.68\%}$$

Investor 2 will select the risk-free asset over the selected portfolio if the risk-free rate was 5.68% or higher.

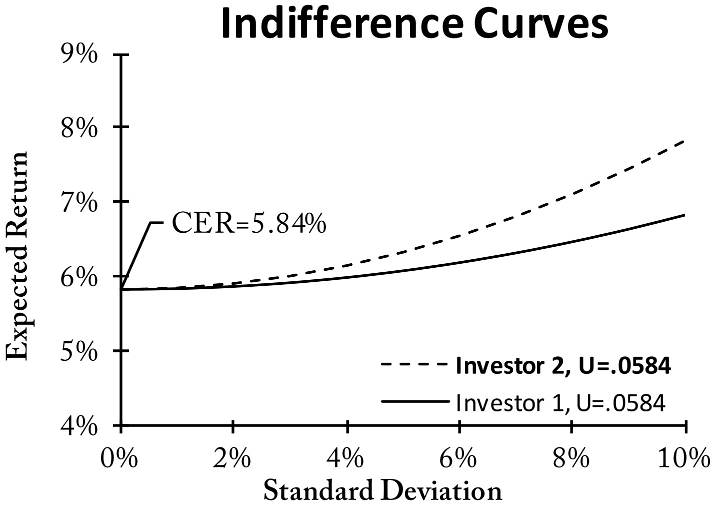
Discussion

The Certainty Equivalent Rate is the rate that a risk-free asset would need to provide to give the same utility to a risk-averse investor as a risky asset. In part c, investor 2 is indifferent between a risk-free asset with an expected return of 5.68% and the low-risk portfolio with an expected return of 6% and standard deviation of 4%.

Indifference Curve Plots

We could also plot the indifference curves for two investors at the *same* level of utility (see below). Note that this is a little different than the graph for the problem.

At the same level of utility, the indifference curve for an investor with a greater degree of risk-aversion will *always* be higher than the indifference curve for an investor with lower risk-aversion, except at the y-axis intercept.



Source

BKM 6

More Practice

CAS 2015 – 1	RF BKM 6 – 1
	RF BKM 6 – 3
	RF BKM 6 – 5
	RF BKM 6 – 6

Capital Allocation Line

BKM 6

Problem

An investor is selecting a portfolio allocated between a balanced mutual fund comprised of 60% stocks and 40% bonds as well as 1-Month T-Bills.

Below are the assumed expected return and standard deviation for each asset:

	Expected Return	Standard Deviation
Mutual Fund	9.6%	20%
1-Month T-Bills	2%	0%

- The borrowing rate is 4%.

The investor allocates 80% of her portfolio to the mutual fund and the rest to 1-Month T-Bills.

a. Calculate the expected return and standard deviation of the complete portfolio.

b. Calculate the Sharpe ratio for the Capital Allocation Line.

c. Plot the Capital Allocation Line and show the point on the curve where the complete portfolio for the investor lies.

Solution Recipe

Part a – Complete Portfolio

1) **Calculate the expected return of the complete portfolio as a weighted average between the expected returns from the risky and risk-free assets. Assume zero correlation between the risky and risk-free asset.**

$$\boxed{\mathrm{E}[r_c]=y\cdot\mathrm{E}[r_p]+(1-y)\cdot r_f}$$

$$\mathrm{E}[r_c]=.8\times.096+(1-.8)\times.02$$

$$=\boxed{.0808}$$

2) **Calculate the standard deviation of the complete portfolio based on the standard deviation of the risky asset and the percentage of the allocation to the risky asset.**

$$\boxed{\sigma_c=y\cdot\sigma_p}$$

$$\sigma_c=.8\times.2$$

$$=\boxed{.16}$$

Part b – Sharpe Ratio

3) **Calculate the Sharpe ratio based on the assumptions of the *risky asset*. This is the slope of the Capital Allocation Line. If there is a borrowing rate greater than the risk free rate, there will be different Sharpe ratios for allocation percentages $y>1$ and $y \leq 1$.**

$$\boxed{S_{y\leq 1}=\frac{\mathrm{E}\left[r_p\right]-r_f}{\sigma_p}}$$

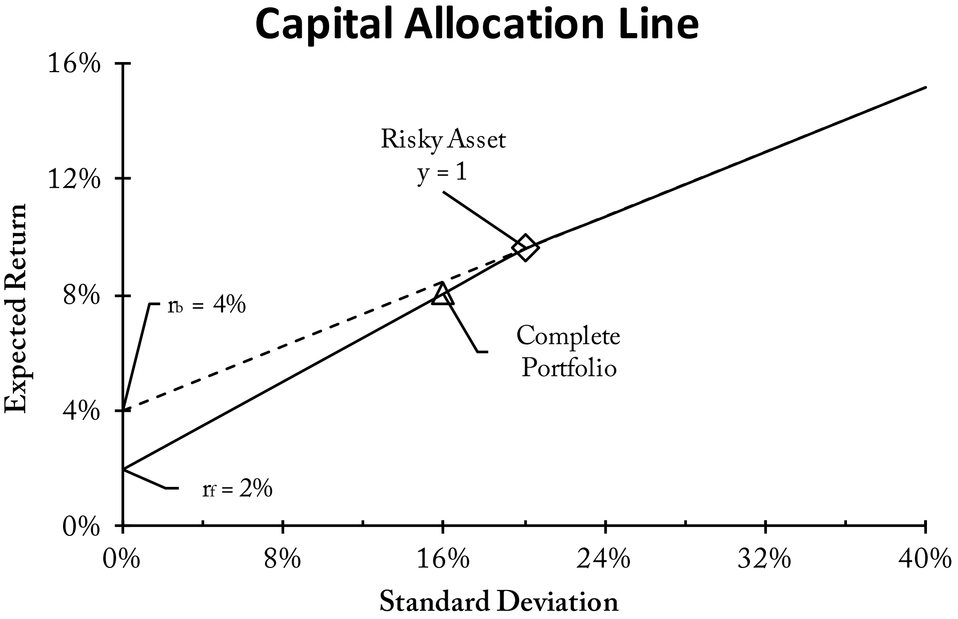
$$\boxed{S_{y>1}=\frac{\mathrm{E}\left[r_p\right]-r_b}{\sigma_p}}$$

$$S_{y\leq 1}=\frac{.096-.02}{.2}=\boxed{.38}$$

$$S_{y>1}\frac{.096-.04}{.2}=\boxed{.28}$$

Part c – Capital Allocation Line

4) **Plot the Capital Allocation Line, showing different levels of expected return and standard deviation for different allocations between the risky and risk-free asset. If there is a borrowing rate, there will be a kink in the CAL at the transition point where $y=1$.**



Discussion

The Sharpe ratio is the reward-to-risk ratio for the risky asset and is the slope of the Capital Allocation Line. The CAL shows all the possible complete portfolios possible between a risky asset and a risk free asset.

Key points on the CAL graph

- CAL *y*-axis intercept is at the risk-free rate. This is where $y = 0$.
- For $y \leq 1$, the slope of the CAL is the Sharpe Ratio at the risk-free rate.
- For $y > 1$, the slope of the CAL is the Sharpe Ratio at the borrowing rate.
- The CAL is kinked at $y = 1$, where the investor transitions from investing at the risk-free rate to borrowing at the borrowing rate.

Source

BKM 6

More Practice

CAS 2014 – 1
CAS 2016 – 1
RF BKM 6 - 2

Optimal Allocation: Risky and Risk-Free Assets

BKM 6

Problem

A risk-averse investor with \$200,000 is selecting a portfolio allocated between a mutual fund tracking a broad index of common stocks and 1-Month T-Bills.

Given the following assumptions:

Index fund expected risk premium	7.8%
Index fund standard deviation	20%
1-Month T-Bill expected return	2.5%
Investor degree of risk-aversion	3.0

a. Select the optimal portfolio for the investor.

b. Graph the Capital Market Line and the indifference curve for this investor at the utility level of the optimal portfolio.

Solution Recipe

Part a – Optimal Portfolio

1) Calculate *y, the optimal allocation to the risky asset that maximizes investor utility.**

$$\boxed{y^* = \frac{\mathrm{E}\left[r_p\right] - r_f}{A\sigma^2}}$$

$$\text{Optimal Risky Asset}\,\% = \frac{\text{Risk Premium}}{A\sigma^2}$$

$$y^* = \frac{.078}{3 \times .20^2}$$

$$= .65$$

2) Allocate the portfolio according to the optimal allocation.

$$\text{Risky Asset Allocation} = y^* \cdot \text{Portfolio Balance}$$

$$\text{Index Fund Allocation} = .65 \times \$200{,}000 = \boxed{\$130{,}000}$$

$$1\text{-Month T-Bills} = \boxed{\$70{,}000}$$

Part b – Capital Market Line and Indifference Curve

3) Calculate the expected return and standard deviation of the optimal portfolio.

$$\boxed{\mathrm{E}\left[r_c\right] = y \cdot \mathrm{E}\left[r_p\right] + (1-y) \cdot r_f}$$

$$\boxed{\sigma_c = y \cdot \sigma_p}$$

$$\mathrm{E}\left[r_c\right] = .65 \times (7.8\% + 2.5\%) + (1 - .65) \times 2.5\% = 7.57\%$$

$$\sigma_c = .65 \times .20 = .13$$

4) **Calculate the utility value of the optimal portfolio, based on the investor's degree of risk aversion.**

$$\boxed{U = E(r) - \tfrac{1}{2}A\sigma^2}$$

$$U = .0757 - \tfrac{1}{2}\times 3\times .13^2$$
$$= .0504$$

5) **Calculate the equation for the indifference curve at the utility value of the optimal portfolio.**

$$U = E(r) - \tfrac{1}{2}A\sigma^2$$

$$.0504 = \mathrm{E}[r] - \tfrac{1}{2}\times 3\times \sigma^2$$
$$\rightarrow \mathrm{E}[r](\sigma) = .0504 + 1.5\times\sigma^2$$

6) **Calculate the equation for the Capital Market Line (CML). The *y*-intercept is the risk-free rate and the slope is the Sharpe ratio.**

$$\boxed{S = \frac{\mathrm{E}[r_p] - r_f}{\sigma_p}}$$

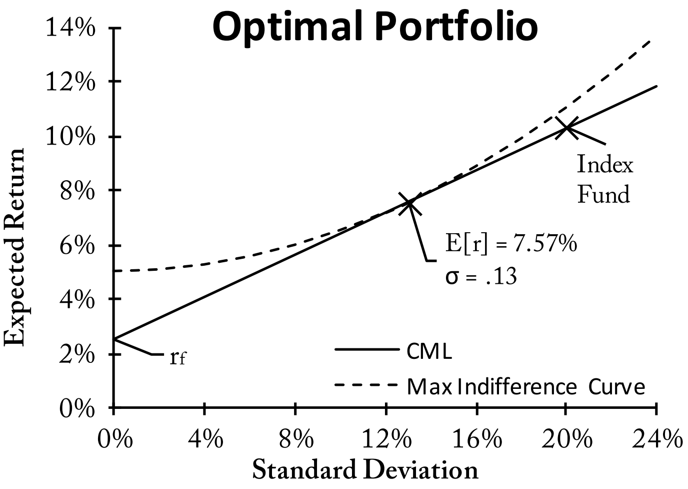
$$S = \frac{.078}{.2} = .39$$

CAL Equation:

$$\mathrm{E}[r](\sigma) = r_f + \frac{\mathrm{E}[r_p] - r_f}{\sigma_p}\cdot\sigma$$

$$\mathrm{E}[r](\sigma) = .025 + .39\times\sigma$$

7) **Plot the CML and indifference curve. At the point of the optimal portfolio, the CML will be tangential to the indifference curve.**



		Expected Return	
y %	Standard Deviation	Indifference Curve	CML
0%	0.00	5.04%	2.50%
10%	0.02	5.10%	3.28%
20%	0.04	5.28%	4.06%
30%	0.06	5.58%	4.84%
40%	0.08	6.00%	5.62%
50%	0.10	6.54%	6.40%
60%	0.12	7.20%	7.18%
65%	***0.13***	***7.57%***	***7.57%***
70%	0.14	7.98%	7.96%
80%	0.16	8.88%	8.74%
90%	0.18	9.90%	9.52%
100%	0.20	11.04%	10.30%
110%	0.22	12.30%	11.08%
120%	0.24	13.68%	11.86%

$\underline{y = .50 \text{ and } \sigma = .10}$

$$\mathrm{E}[r]_{Indifference} = .0504 + 1.5\times .10^2 = .0654$$

$$\mathrm{E}[r]_{CML} = .025 + .39\times 10 = 0.064$$

Discussion

Since we're using a broad index fund of equities as the risky asset, the CAL is also the Capital Market Line. The CML shows all the possible portfolio allocations between 100% in the risk-free asset and 100%+ in the risky asset.

The goal is to find the portfolio along the CML that maximizes the investor's utility. This is the point on the CML that is tangential to the indifference curve with the highest utility value possible for the investor along the CML.

A possible twist to this problem is that you might be given a different utility function. In this case, you need to take the derivative with respect to *y*, the allocation % to the risky asset. Then, solve for y^* by setting the derivative to zero and solving for *y*.

Source

BKM 6

More Practice

CAS 2018 – 1c
CAS 2016 – 1
CAS 2013 – 1
CAS 2012 – 1
CAS 2011 – 1
RF BKM 6 – 4
RF BKM 6 - 5

Optimal Complete Portfolio

BKM 7

Problem

An investor is selecting a portfolio comprised of a stock mutual fund, bond mutual fund and T-Bills. Given the following information:

Asset	Expected Return	Standard Deviation
Stock Fund	12%	20%
Bond Fund	5%	11%
T-Bills	2%	0%

- Correlation coefficient between Stock and Bond Funds 0.30
- Investor degree of risk aversion 4.0

a. Calculate the expected return and standard deviation of the optimal risky portfolio.

b. Calculate the optimal complete portfolio for the investor and the expected return and standard deviation of the optimal complete portfolio.

Solution Recipe

Part a – Optimal Risky Portfolio of Two Risky Assets

1) Calculate E[R], the expected excess return for each risky asset.

$$\boxed{\mathrm{E}[R]=\mathrm{E}[r]-r_f}$$

$$\mathrm{E}[R_D]=.05-.02=.03$$

$$\mathrm{E}[R_E]=.10$$

2) Calculate the weights of the optimal risky portfolio that maximizes the Sharpe ratio. This results in the steepest CAL and maximizes the reward-to-risk.

$$\boxed{w_D=\frac{\mathrm{E}[R_D]\sigma_E^2-\mathrm{E}[R_E]\sigma_{DE}}{\mathrm{E}[R_D]\sigma_E^2+\mathrm{E}[R_E]\sigma_D^2-(\mathrm{E}[R_D]+\mathrm{E}[R_E])\sigma_{DE}}}$$

$$\boxed{w_E=1-w_D}$$

$$w_D=\frac{.03\times.20^2-.10\times(.20\times.11\times.30)}{.03\times.20^2+.10\times.11^2-(.03+.10)\times(.20\times.11\times.30)}$$

$$=.348$$

$$w_E=1-.348=.652$$

3) Calculate the expected return and standard deviation for the optimal risky portfolio.

$$\boxed{\mathrm{E}\left[r_p\right]=\sum w_i\cdot\mathrm{E}\left[r_i\right]}$$

2-Assets:

$$\boxed{\sigma_p^2=\sum\sum w_i w_j\,\mathrm{Cov}\left(r_i,r_j\right)}$$

$$\boxed{\sigma_p^2=w_D^2\sigma_D^2+w_E^2\sigma_E^2+2w_D w_E\,\mathrm{Cov}\left(r_D,r_E\right)}$$

$$\mathrm{E}\left[r_p\right]=.348\times.05+.652\times.12=\boxed{.0956}$$

$$\sigma_p^2=.348^2\times.11^2+.652^2\times.20^2+2\times.348\times.652\times(.20\times.11\times.30)=.0215$$

$$\rightarrow\sigma_p=\sqrt{.0215}=\boxed{.1465}$$

Part b – Optimal Complete Portfolio for an Investor

4) Calculate the allocation to the optimal risky portfolio, based on the investor's degree of risk aversion.

$$\boxed{y^*=\frac{\mathrm{E}\left[r_p\right]-r_f}{A\sigma^2}}$$

$$\textit{Optimal Risky Asset}\,\%=\frac{\textit{Risk Premium}}{A\sigma^2}$$

$$y^*=\frac{.0956-.02}{4\times.1465^2}$$

$$=.881$$

5) Calculate the asset allocation between the risky assets and risk-free asset.

$$\boxed{\begin{aligned}&\textit{allocation}_D=y^*\cdot w_D\\&\textit{allocation}_{\textit{risk-free}}=1-y^*\end{aligned}}$$

$$\textit{allocation}_D=.881\times.348=.307$$

$$\textit{allocation}_E=.881\times(1-.348)=.574$$

$$\textit{allocation}_{\textit{risk-free}}=1-.881=.119$$

6) Calculate the expected return and standard deviation of the optimal complete portfolio.

$$\boxed{\mathrm{E}\left[r_c\right]=y\cdot\mathrm{E}\left[r_p\right]+(1-y)\cdot r_f}$$

$$\boxed{\sigma_c=y\cdot\sigma_p}$$

$$\mathrm{E}\left[r_c\right]=.881\times.0956+(1-.881)\times.02=.0866$$

$$\sigma_c=.881\times.1465=.129$$

Discussion

Minimum Variance Portfolio for Two Assets

We could also solve for the weights that minimize the variance of a risky portfolio (as opposed to maximizing the Sharpe ratio). The minimum variance portfolio has the following weights:

$$\boxed{w_{\min,A}=\frac{\sigma_B^2-\sigma_{AB}}{\sigma_A^2+\sigma_B^2-2\sigma_{AB}}}\qquad\boxed{w_{\min,B}=1-w_{\min,A}}$$

For assets that are perfectly positively correlated with $\rho = 1$, there's no diversification benefit and the minimum variance portfolio is simply the asset with the lowest variance.

For assets that are perfectly negatively correlated with $\rho = -1$, the minimum variance portfolio will have a portfolio variance of zero:

$$w_{\min,A} = \frac{\sigma_B}{\sigma_A + \sigma_B} \qquad \text{for } \rho = -1$$

Separation Property

The key concept in this problem is the **Separation Property**. An investment manager will separate portfolio selection for investors into two independent tasks (shown in parts a and b):

1. First, in part a, we find the **optimal risky portfolio** on the efficient frontier that will maximize the Sharpe ratio and provide the steepest CAL. This optimal risky portfolio would be used for all investors.

   Below, you can see that the *optimal risky portfolio* has the highest Sharpe Ratio:

Asset	Sharpe Ratio
Stock Fund	0.5
Bond Fund	0.273
Optimal Risky	**0.516**

   $$S_{Bond} = \frac{.05 - .02}{.11} = .273$$

   $$S_{Optimal\ Risky} = \frac{.0956 - .02}{.1465} = .516$$

2. Then, in part b, we calculate the **optimal complete portfolio** specific to an investor by allocating between the optimal risky portfolio and the risk-free asset, using the investor's degree of risk-aversion. The result is a portfolio that maximizes utility *for the investor*.

   Below, you can see that the *optimal complete portfolio* has the highest utility for an investor with $A = 4$:

Asset	Utility ($A = 4$)
T-Bills	0.02
Optimal Risky	0.0523
Optimal Complete	**0.0533**

   $$U_{Optimal\ Complete} = .0866 - \tfrac{1}{2} \times 4 \times .129^2 = .0533$$

Investors with *higher* degrees of risk aversion will allocate *less* to the optimal risky portfolio than investors with lower degrees of risk aversion.

Source

BKM 7

More Practice

CAS 2019 – 2	RF BKM 7 – 1
CAS 2017 – 3	RF BKM 7 – 2
CAS 2017 – 4	RF BKM 7 – 5
CAS 2016 – 3	RF BKM 7 – 6
CAS 2015 – 1	RF BKM 7 – 7
CAS 2014 – 5	RF BKM 7 – 8

Diversification in an Equally-Weighted Portfolio

BKM 7

Problem

Given the following information for a universe of available risky securities:

- Returns are identically distributed with an expected return of 10% and standard deviation of 50%
- Securities share a common correlation of 0.45

a. For an equally-weighted portfolio of 12 stocks, calculate the firm-specific and systematic risk.

b. Calculate the minimum number of stocks necessary so that the standard deviation is within 0.5% points of the standard deviation of a well-diversified portfolio that maximizes the reward-to-risk ratio.

Solution Recipe

Part a – Firm-Specific and Systematic Risk

1) Calculate the total variance of the portfolio.

$$\boxed{\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{COV}}$$

$$\sigma_p^2 = \frac{1}{12}\times .5^2 + \frac{12-1}{12}\times .5^2 \times .45$$

$$= .1240$$

2) Calculate the systematic risk, the total variance of an equally weighted portfolio as $n \to \infty$ (See the note in the discussion).

$$\boxed{\textit{systematic risk} = \overline{COV}}$$

$$\textit{systematic risk} = \sigma_i \sigma_j \rho$$

$$\textit{systematic risk} = .5^2 \times .45$$

$$= \boxed{.1125}$$

3) Calculate the firm-specific risk for the portfolio by backing out the systematic risk from the total risk.

$$\boxed{\sigma_p^2 = \textit{systematic risk} + \textit{firm-specific risk}}$$

$$\textit{firm-specific risk} = .1240 - .1125$$

$$= \boxed{.0115}$$

Part b – Minimum Number of Securities for a Target Standard Deviation

1) Calculate the standard deviation target.

$$\sigma_p = \sqrt{\textit{systematic risk}} + \textit{target}$$

$$\sigma_p = \sqrt{.1125} + .005$$

$$= .3404$$

2) Solve for the minimum number of stocks needed so that the portfolio standard deviation is less than the target.

$$\boxed{\sigma^2_{target} > \sigma^2_p = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{COV}}$$

$$.3404^2 \;\; > \frac{1}{n}.5^2 + \frac{n-1}{n}.5^2 \times .45$$

$$.1159n \;\; > .25 + .1125n - .1125$$

$$.00338n > .1375$$

$$n \qquad > 40.7$$

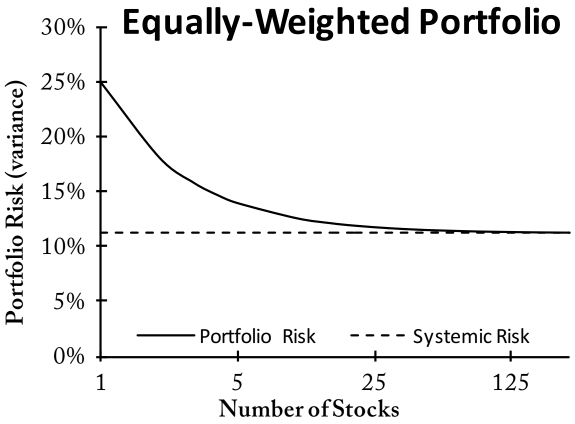
At least 41 stocks are needed in the portfolio.

Discussion

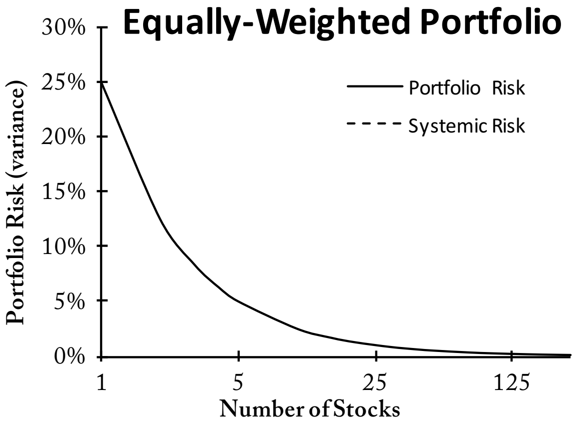
This is more of a hypothetical exercise than how a real stock portfolio works. The point is to demonstrate the benefit of diversification in a portfolio. Even when the underlying stocks are positively correlated, adding additional stocks will lower the portfolio variance down towards the average covariance between stocks. The average covariance between stocks is the systematic risk that is undiversifiable.

If securities are uncorrelated, then the risk can be completely diversified away as the number of securities in the portfolio increases without bound. This is shown in the graphs below:

$\rho = .45$



$\rho = 0$



Note on systematic risk:
Based on BKM 7 (see Concept Check 7.5), systematic risk is the average covariance, the limit as $n \to \infty$. Remember, the systematic risk shouldn't change based on how many stocks a portfolio has.

Source

BKM 7

More Practice

CAS 2015 – 2 RF BKM 7 – 3

Single Index Model for Portfolio

BKM 8

Problem

Given the following security analysis of a two-stock portfolio:

Stock	Alpha	Beta	Standard Deviation	% of Portfolio
A	1.2%	0.9	37.5%	60%
B	0.5%	1.5	45%	40%

Expected return of the market index	12%
Standard deviation of the market index	25%
Risk free rate	3%

a. Calculate the expected return for the portfolio based on the single index model.

b. Calculate the covariance and correlation between stocks A and B.

c. Calculate the total portfolio variance and firm-specific portion of the portfolio variance.

Solution Recipe

Part a – Single Index Model Portfolio

1) **Calculate α_p as a weighted average of the stocks' α_i s.**

$$\boxed{\alpha_p = \sum w_i \alpha_i}$$

$$\alpha_p = .6 \times 1.2\% + .4 \times 0.5\%$$
$$= 0.92\%$$

2) **Calculate β_p as a weighted average of the stocks' β_i s.**

$$\boxed{\beta_p = \sum w_i \beta_i}$$

$$\beta_p = .6 \times 0.9 + .4 \times 1.5$$
$$= 1.14$$

3) **Calculate the expected *risk premium* of the portfolio using the single index model equation.**

$$\boxed{\mathrm{E}\left[R_p\right] = \alpha_p + \beta_p \cdot \mathrm{E}\left[R_m\right]}$$

$$\mathrm{E}\left[R_p\right] = 0.92\% + 1.14 \times (12\% - 3\%)$$
$$= 11.18\%$$

4) **Calculate the expected return of the portfolio by adding r_f to the expected risk premium.**

$$\boxed{\mathrm{E}\left[r_p\right]=\mathrm{E}\left[R_p\right]+r_f}$$

$$\mathrm{E}\left[r_p\right]=11.18\%+3\%$$
$$=\boxed{14.18\%}$$

Part b – Covariance Between Stocks: Single Index Model

5) **Calculate the covariance between two stocks based on their relationship to the market index.**

$$\boxed{\mathrm{Cov}\left(r_i,r_j\right)=\beta_i\beta_j\sigma_m^2}$$

$$\mathrm{Cov}\left(r_i,r_j\right)=0.9\times1.5\times.25^2$$
$$=\boxed{0.084375}$$

6) **Calculate the coefficient of correlation between two stocks.**

$$\boxed{\mathrm{Corr}\left(r_i,r_j\right)=\frac{\beta_i\beta_j\sigma_m^2}{\sigma_i\sigma_j}}$$

$$\mathrm{Corr}\left(r_i,r_j\right)=\frac{0.084375}{.375\times.45}$$
$$=\boxed{.50}$$

Part c – Systematic and Firm-Specific Portfolio Variance

7) **Calculate $\sigma^2(e_i)$, the firm-specific variance, for each stock.**

$$\boxed{\sigma_i^2=\beta_i^2\sigma_m^2+\sigma^2(e_i)}$$

$$.375^2=0.9^2\times.25^2+\sigma^2(e_A) \qquad \rightarrow\sigma^2(e_A)=.09$$
$$.45^2=1.5^2\times.25^2+\sigma^2(e_B) \qquad \rightarrow\sigma^2(e_B)=.06188$$

8) **Calculate the portfolio's firm-specific variance as a weighted sum of the underlying stocks' firm-specific variance. Make sure to *__square the weights__*.**

$$\boxed{\sigma^2(e_p)=\sum w_i^2\sigma^2(e_i)}$$

$$\sigma^2(e_p)=.6^2\times.09+.4^2\times.06188$$
$$=\boxed{.0423}$$

9) **Calculate the portfolio's variance as a sum of the systematic and firm-specific variance.**

$$\sigma_p^2=\beta_p^2\sigma_m^2+\sigma^2(e_p)$$

$$\sigma_p^2=1.14^2\times.25^2+.0423$$
$$=\boxed{0.1235}$$

Discussion

Watch out for the difference between the ***total*** variance, σ_i^2, and the ***firm-specific*** variance, $\sigma^2(e_i)$. An exam problem could give either of these and that will impact how you need to solve the problem.

Pros & Cons of Single-Index Model vs. Markowitz

Advantages	Disadvantages
• The number of parameters is significantly lower than Markowitz. ○ $3n + 2$ estimates are needed*	• It oversimplifies uncertainty between macro and micro risk and doesn't allow for industry-specific effects.
• It allows for specialization in security analysis since covariance is based on the relationships to the market index.	• The optimal portfolio from the single-index model can be inferior to the optimal portfolio from the Markowitz model.

* These are the estimates needed:

- n estimates of α_i
- n estimates of β_i
- n estimates of σ_i
- 1 estimate for $E[R_M]$
- 1 estimate for σ_M^2

Shortcut to a 2-stock portfolio total portfolio variance

If we don't need the breakout between the systematic and firm-specific variance, we can calculate the total portfolio variance more directly based on the weights and the covariance.

$$\boxed{\sigma_p^2 = w_i^2\sigma_i^2 + w_j^2\sigma_j^2 + 2w_iw_j\,\mathrm{Cov}\left(r_i, r_j\right)}$$

$$\sigma_p^2 = .6^2 \times .375^2 + .4^2 \times .45^2 + 2 \times .6 \times .4 \times \left(.9 \times 1.5 \times .25^2\right)$$

$$= \boxed{0.1235}$$

Source

BKM 8

More Practice

CAS 2019 – 3
CAS 2015 – 3
RF BKM 8 – 1

Optimal Risky Portfolio - Single Index Model

BKM 8

Problem

A portfolio manager actively researched three stocks to use in an active portfolio. The active portfolio complements a passive portfolio based on the S&P 500 index. Below is the input list and market assumptions:

• S&P 500 Expected return	10%
• S&P 500 Standard deviation	20%
• Risk-free rate	2%

Stock	Alpha	Beta	Residual Variance
A	1%	0.8	8%
B	3%	1.5	9%
C	-2%	1.2	8%

The portfolio manager can take short positions in securities.

a. Calculate the weights of the optimal risky portfolio based on the index model.

b. Calculate the expected risk premium and variance of the optimal risky portfolio.

c. Calculate the Sharpe Ratio of the optimal risky portfolio.

Solution Recipe

Part a – Optimal Risky Portfolio Weights

1) Calculate the initial weight of each stock in the active portfolio, based on its contributed α_i and *residual variance*, $\sigma^2(e_i)$.

$$\boxed{w_i^0 = \frac{\alpha_i}{\sigma^2(e_i)}}$$

$$w_a^0 = \frac{.01}{.08} = .125 \qquad w_b^0 = .333 \qquad w_c^0 = -.25$$

2) Scale the initial weights so they sum to 1.

$$\boxed{w_i = \frac{w_i^0}{\sum w_i^0}}$$

$$w_a = \frac{.125}{.125 + .333 - .25} = .6 \qquad w_b = 1.6 \qquad w_c = -1.2$$

3) Calculate the weighted α_A of the active portfolio.

$$\boxed{\alpha_A = \sum w_i \alpha_i}$$

$$\alpha_A = .6 \times .01 + 1.6 \times .03 + (-1.2) \times (-.02) = .078$$

4) Calculate the residual variance of the active portfolio. Make sure to *square the weights*.

$$\boxed{\sigma^2(e_A)=\sum w_i^2\sigma^2(e_i)}$$

$$\sigma^2(e_A)=.6^2\times.08+1.6^2\times.09+(-1.2)^2\times.08$$

$$=.3744$$

5) Calculate the weighted β_A of the active portfolio.

$$\boxed{\beta_A=\sum w_i\beta_i}$$

$$\beta_A=.6\times.8+1.6\times1.5+(-1.2)\times1.2$$

$$=1.44$$

6) Calculate the initial weight of the active portfolio.

$$\boxed{w_A^0=\frac{\alpha_A/\sigma^2(e_A)}{\mathrm{E}[R_M]/\sigma_M^2}}$$

$$w_A^0=\frac{.078/.3744}{(.10-.02)/.20^2}=.10417$$

7) Calculate the final weight of the active portfolio by adjusting for β_A.

$$\boxed{w_A^*=\frac{w_A^0}{1+(1-\beta_A)w_A^0}}$$

$$w_A^*=\frac{.10417}{1+(1-1.44)\times.10417}=.10917$$

8) Calculate the weights of the optimal risky portfolio, including the passive portfolio and each security in the active portfolio.

$$\boxed{\begin{aligned}w_M^*&=1-w_A^*\\ w_i^*&=w_A^*\cdot w_i\end{aligned}}$$

$$w_M^*=1-.10917$$

$$=.8908$$

$$w_a^*=.10917\times.6=.0655$$

Asset	Weight
S&P 500	.8908
Stock A	.0655
Stock B	.1747
Stock C	-.1310

Part b – Expected Return and Variance of a Portfolio

9) Calculate the expected risk premium for the portfolio. This calculation uses a weighted alpha and beta based on the active and passive portfolio.

$$\boxed{\mathrm{E}[R_P]=w_A^*\alpha_A+(w_M^*+w_A^*\beta_A)\mathrm{E}[R_M]}$$

$$\mathrm{E}[R_P]=.10917\times.078+(.8908+.10917\times1.44)\times(.10-.02)$$

$$=\boxed{.0924}$$

10) Calculate the variance of the optimal risky portfolio.

$$\boxed{\sigma_p^2=(w_M^*+w_A^*\beta_A)^2\sigma_M^2+w_A^2\sigma(e_A)^2}$$

$$\sigma_p^2=(.8908+.10917\times1.44)^2\times.2^2+.10917^2\times.3744$$

$$=\boxed{.0484}$$

Part c – Sharpe Ratio of an Optimal Risky Portfolio

11) Calculate the Sharpe ratio based on the expected risk premium and standard deviation *OR* by adjusting the Sharpe ratio of the market upward due to the information ratio of the active portfolio.

$$\boxed{S_p = \frac{\mathrm{E}\left[R_p\right]}{\sigma_p}}$$

$$S_p = \frac{.0924}{\sqrt{.0484}} = \boxed{.420}$$

Alternative Calculation

$$\boxed{S_p^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma\left(e_A\right)}\right]^2}$$

$$Information\ Ratio = \frac{\alpha_A}{\sigma\left(e_A\right)}$$

$$S_p^2 = \left(\frac{.10-.02}{.20}\right)^2 + \left[\frac{.078}{\sqrt{.3744}}\right]^2 = .17625$$

$$\rightarrow S_p = \boxed{.420}$$

Discussion

What we're doing here is using the index portfolio as the base of the optimal risky portfolio. Then, using the active portfolio, we "tilt" the optimal risky portfolio towards individual securities with positive alphas and away from securities with negative alphas. The Sharpe ratio of the new optimal risky portfolio will be increased by the positive *information ratio* and be higher than the Sharpe ratio of the passive index portfolio.

Once we have the optimal risky portfolio, we can use the "Optimal Complete Portfolio" recipe to find the optimal complete portfolio for a specific investor based on the investor's degree of risk aversion. The complete portfolio will be some combination of the optimal risky portfolio and a risk-free asset (or risk-free borrowing).

Possible Twists

- If a problem says shorting securities isn't allowable, just ignore all the stocks with a negative alpha. In this problem, we would ignore Stock C from the start.
- If given the actual variance for each stock, σ_i^2, first solve for each stock's residual variance, $\sigma^2(e_i)$.

Source

BKM 8

More Practice

CAS 2019 – 3
CAS 2013 – 3
RF BKM 8 – 2
RF BKM 8 – 3

Myers-Read Allocation

Cummins Capital & Goldfarb

Problem

Given the following information for an insurer allocating capital to three lines of business:

- Total capital for the insurer $285M
- Overall volatility parameter, σ 0.20

The value of the insolvency put per dollar of total liabilities, p, has the following rate of change assumptions:

- Partial derivative of p with respect to s, the surplus-to-liability ratio -0.17
- Partial derivative of p with respect to σ , the volatility parameter 0.11

Business	Covariance of Loss$_i$ and total Loss, σ_{iL}	Covariance of Loss$_i$ and the firm's asset portfolio, σ_{iV}
1	0.32	0
2	0.25	0
3	0.15	0

- Volatility parameter of total losses L, σ_L^2 0.224
- Assume no covariance between the insurer's assets and losses

The insurer has the following loss liabilities by line of business:

	Business 1	Business 2	Business 3
Loss Liability	80M	60M	125M

Calculate the Myers-Read capital allocation to the lines of business.

Solution Recipe

1) Calculate the surplus-to-liability ratio, s_i , for each line of business.

$$\boxed{s_i = s - \frac{\partial p/\partial\sigma}{\partial p/\partial s}\cdot\frac{\left[\left(\sigma_{iL}-\sigma_L^2\right)-\left(\sigma_{iV}-\sigma_{LV}\right)\right]}{\sigma}}$$

$$s_i = \frac{Capital_i}{Loss_i}$$

$$p = \frac{Insolvency\ Put}{\sum Loss_i}$$

$$s_1 = \frac{285\text{M}}{265\text{M}} - \frac{.11}{-.17}\times\frac{\left[(.32-.224)-(0-0)\right]}{.20} = 1.386$$

$$s_2 = \frac{285\text{M}}{265\text{M}} - \frac{.11}{-.17}\times\frac{\left[(.25-.224)-(0-0)\right]}{.20} = 1.160$$

$$s_3 = \frac{285\text{M}}{265\text{M}} - \frac{.11}{-.17}\times\frac{\left[(.15-.224)-(0-0)\right]}{.20} = 0.836$$

2) **Calculate the capital required for each line of business by applying the surplus-to-liability leverage ratio to the loss liability for the line.**

$$\boxed{s_i = \frac{Capital_i}{Loss_i}}$$

$$1.386 = \frac{Capital_1}{80\text{M}} \quad \rightarrow Capital_1 = 110.9\text{M}$$

$$1.160 = \frac{Capital_2}{60\text{M}} \quad \rightarrow Capital_2 = 69.6\text{M}$$

$$0.836 = \frac{Capital_3}{125\text{M}} \quad \rightarrow Capital_3 = 104.5\text{M}$$

Note: Notice that the capital allocations by line sum to the total capital allocation for the insurer, 285M.

Discussion

For the Myers-Read method, capital is allocated by looking at the marginal impact of very small changes to the loss liabilities of a line of business on the overall capital need for the insurer. This is different than Merton-Perold which allocates capital according to the marginal impact of adding the *entire business* to the insurer.

Unlike the M-P method, the Myers-Read method allocates 100% of the insurer's capital so the allocation of capital is different. Because of this, Cummins mentions that the M-P method may be more appropriate if we're adding an entire business to the insurer. The M-R method may be more appropriate for "normal operations" of the insurer when changes to the business are small and along the margins.

Impact of Covariances on the Allocation

- When the covariance between the **loss of line *i* and the insurer's total loss is higher** (larger σ_{iL}), the surplus allocation is **higher**.
    - When line *i* has higher losses, it's more likely that the total losses will be greater, meaning the line is relatively riskier for the firm and a greater capital allocation is needed.
- When the covariance between the **loss of line *i* and the insurer's asset portfolio is higher** (larger σ_{iV}), the surplus allocation is **lower**.
    - When line *i* has higher losses, it's more likely that the asset portfolio will be greater, creating a natural hedge.

Source

Cummins Capital – pg. 23-25
Goldfarb – pg. 50-53

Note: The Goldfarb paper adds more context about the Myers-Read method, so make sure to check it out and the example there.

More Practice

RF Goldfarb – 6

Capital Allocation by Percentile Layer

Bodoff

Problem

Given the following possible loss scenarios for an insurer by line of business (LOB):

Loss Scenario	Total Loss ($M)	LOB A Loss ($M)	LOB B Loss ($M)	Probability
1	180	120	60	.01
2	80	80	0	.04
3	60	0	60	.10
4	30	0	30	.35
5	0	0	0	.50

- The insurer uses VaR 99% as the capital requirement.

Allocate the required capital to line of business using the following methods:

a. coVaR
b. Alternative coVaR
c. coTVaR
d. Allocation by Percentile Layer

Solution Recipe

Part a – coVaR Method

1) Calculate the required capital for the firm, based on VaR.

$$\boxed{VaR_{x\%} = \text{Net Loss at } x\% \text{ percentile}}$$

$$Capital = VaR_{99\%} = 80\text{M} \qquad \rightarrow \text{Loss Scenario } 2$$

2) Allocate the capital to the LOBs in proportion to their contributed loss to the VaR loss scenario.

$$LOB\,Capital_A^{coVaR} = 80\text{M} \times \frac{80}{80} = 80\text{M}$$

$$LOB\,Capital_B^{coVaR} = 0\text{M}$$

Part b – Alternative coVaR

3) Allocate capital to each event scenario that's greater than or equal to the VaR loss scenario in proportion to that event's probability.

$$\boxed{Event\,Capital_i^{alt\,coVaR} = Capital \cdot \frac{prob_i}{\sum\limits_{Losses \geq VaR} prob_i}}$$

$$Event\,Capital_1 = 80\text{M} \times \frac{.01}{.01 + .04} = 16\text{M}$$

$$Event\,Capital_2 = 80\text{M} \times \frac{.04}{.01 + .04} = 64\text{M}$$

4) **Sub-allocate each event's capital to the lines of business in proportion to the contributed loss of the line of business. Sum the capital by line of business over all events to get the total allocation.**

$$\boxed{LOB\,Capital_{i,LOB} = Event\,Capital_i \cdot \frac{Loss_{LOB}}{Loss\ \ for\ Event\, i}}$$

$$LOB\,Capital_{1,LOB\,A} = 16\text{M} \times \frac{120}{180} = 10.67\text{M}$$

Scenario	Capital	LOB A Capital	LOB B Capital
1	16M	10.67M	5.33M
2	64M	64M	0
		74.67M	5.33M

Part c –coTVaR

5) **Allocate capital to each event scenario that's greater than or equal to the VaR loss scenario in proportion to the probability times the loss for the scenario.**

$$\boxed{Event\,Capital_i^{coTVaR} = Capital \cdot \frac{Loss_i \cdot prob_i}{\sum_{Losses \geq VaR} Loss_i \cdot prob_i}}$$

$$Event\,Capital_1 = 80 \times \frac{.01 \times 180}{.01 \times 180 + .04 \times 80} = 28.8\text{M}$$

$$Event\,Capital_2 = 80 \times \frac{.04 \times 80}{.01 \times 180 + .04 \times 80} = 51.2\text{M}$$

6) **Sub-allocate each event's capital to the lines of business in proportion to the contributed loss of the line of business. Sum the capital by line of business over all events to get the total allocation.**

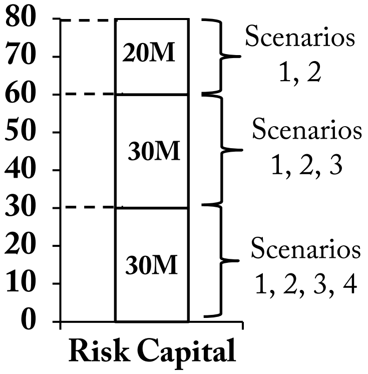
$$\boxed{LOB\,Capital_{i,LOB} = Event\,Capital_i \cdot \frac{Loss_{LOB}}{Loss\ \ for\ Event\, i}}$$

$$LOB\,Capital_{1,LOB\,A} = 28.8\text{M} \times \frac{120}{180} = 19.2\text{M}$$

Scenario	Capital	LOB A Capital	LOB B Capital
1	28.8M	19.2M	9.6M
2	51.2M	51.2M	0
		70.4M	9.6M

Part d –Allocation by Percentile Layer

7) **Split the total risk capital into layers of capital from 0 to the required capital (80M here). The end points of each layer should correspond to the amount of loss increase between events. Note which loss scenarios would exhaust each layer of capital.**



8) For each layer of capital, calculate the conditional exceedance probability for each event that penetrates and fully exhausts the layer of capital. The sum of the CEPs for each layer will be 1.

$$\boxed{CEP_{i,layer} = \frac{\Pr\left(event\ i\right)}{\Pr\left(All\ events\ penetrating\ layer\right)}}$$

$$CEP_{1,\text{0-30M}} = \frac{.01}{.01+.04+.10+.35} = .02$$

$$CEP_{1,\text{30-60M}} = \frac{.01}{.01+.04+.10} = .067$$

$$CEP_{1,\text{60-80M}} = \frac{.01}{.01+.04} = .20$$

CEP by Layer

Scenario	0-30M	30-60M	60-80M
1	.02	.067	.20
2	.08	.267	.80
3	.2	.667	-
4	.7	-	-
Total	1.00	1.00	1.00

9) Allocate the capital for each layer to the events that penetrate the layer by multiplying the CEP for the event times the amount ("width") of the capital layer. Sum up the allocation for an event across the capital layers to get the total capital allocation to the loss scenario.

$$\boxed{Capital_{i,layer} = CEP_{i,layer} \cdot Capital_{layer}}$$

$$\boxed{AC(Loss_i) = \sum Capital_{i,layer}}$$

$$Capital_{1,\text{0-30M}} = .02 \times 30\text{M} = .6\text{M}$$

$$Capital_{1,\text{30-60M}} = .067 \times 30\text{M} = 2\text{M}$$

$$Capital_{1,\text{60-80M}} = .20 \times 20\text{M} = 4\text{M}$$

Capital Allocation by Layer

Scenario	0-30M	30-60M	60-80M	Total
1	0.6	2.0	4.0	6.6
2	2.4	8.0	16.0	26.4
3	6.0	20.0		26.0
4	21.0			21.0
$\text{Capital}_{\text{layer}}$	30	30	20	80

10) Sub-allocate each event's capital to the lines of business in proportion to the contributed loss of the line of business. Sum the capital by line of business over all events to get the total allocation.

$$\boxed{LOB\ Capital_{i,LOB} = AC\left(Loss_i\right) \cdot \frac{Loss_{LOB}}{Loss_i}}$$

$$LOB\ Capital_{i,LOB} = Event\ Capital_i \cdot \frac{Loss_{LOB}}{Loss\ \ for\ Event\ i}$$

$$LOB\ Capital_{1,LOB\ A} = 6.6\text{M} \times \frac{120}{180} = 4.4\text{M}$$

Scenario	Capital	LOB A Capital	LOB B Capital
1	6.6M	4.4M	2.2M
2	26.4M	26.4M	-
3	26.0M	-	26.0
4	21.0M	-	21.0
	80M	**30.8M**	**49.2M**

Discussion

Below is a summary of the capital allocation between lines A and B for the various methods:

Capital Allocation Method	LOB A Capital	LOB B Capital
coVaR	80	0
alt coVaR	74.67	5.33
coTVaR	70.4	9.6
Percentile Layer	30.8	49.2

Losses from LOB A dominate the larges loss scenarios, so under the tail-based allocation methods (coVaR, alternative coVaR, and coTVaR), the vast majority of the capital is allocated to LOB A even though LOB B causes some significant losses with a higher probability.

A drawback of the tail-based methods is that they ignore loss scenarios below the tail threshold.

The percentile layer method allocates a more appropriate amount of capital LOB B.

Cost of Capital Allocation and Reflecting the Premium Offset

In the Bodoff paper (section 7.2.1), he mentions that we need to reflect the premium (net of expenses) when allocating the cost of capital for a line of business. This is because the premium is a contribution to capital and offsets the allocated capital.

To do this, Bodoff says you "**should subtract the collected premium net of expenses from the allocated capital before multiplying by the cost of capital rate.**" Based on this section and formulas 8.0 and 8.1 in Bodoff, my interpretation is that we would adjust the capital like this for a line of business:

$$\begin{aligned} Capital_{PremOffset,LOB} &= Capital_{LOB} - \left(Prem_{LOB} - Expenses_{LOB}\right) \\ &= allocated\ capital - contributed\ capital \end{aligned}$$

CAS Sample IQ #3 – Capital Allocation with Premium Offset

CAS Sample IQ #3 includes an example of this, but with a formula that's an extension of both the Bodoff and Panning papers: $S = L/(1+y) - P(1-e)$.

This formula subtracts of the premium net of expenses: $P(1-e)$. It also discounts the loss-capital amount (L) because the IQ problem states that losses are paid on Dec 31 and Premium is collected on Jan 1.

Source

Bodoff – pg. 3-8

More Practice

CAS 2019 – 13
CAS 2018 – 15
CAS 2017 – 17

RF Bodoff – 1
RF Bodoff – 2
RF Bodoff – 3

RF Bodoff – 3
RF Bodoff – 6
RF Bodoff – 7